Path Integration for the Plane Pendulum with Finite Amplitude

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Abstract

Exact path integration for the one dimensional potential $V = b^2 \cos 2q$ which describes the finite amplitude pendulum is presented.

1. Introduction

In last two decades almost all of the quantum mechanical potential problems which are solvable by the Schrödinger equation method have also been solved by path integrals [1].

In this note we add the finite amplitude plane pendulum potential $V(q) = b^2 \cos 2q$ to the list of the exact path integral solutions.

2. Path Integral for the potential $V = b^2 \cos 2q$

The Lagrangian path integral for the particle of the unit mass moving in the potential

$$V = b^2 \cos(2q), \qquad b^2 \geq 0, \quad q \in [0, 2\pi) \tag{1}$$

is given by

$$K(q, q', T) = \int \mathcal{D}q \exp\left(i \int_0^T [\frac{1}{2}(\dot{q})^2 - b^2 \cos(2q)]\right) \tag{2}$$

represents the probability amplitude for traveling from a point q to q' in the time interval T. The explicit form of the expression (2) is given by the usual time graded form as [2]

$$K(q, q', T) = \lim_{n\to\infty} \left(\frac{1}{2\pi i\varepsilon}\right)^{\frac{n+1}{2}} \left(\prod_{j=1}^{n} \int_0^{2\pi} dq_j\right) \prod_{j=0}^{n} K_j \tag{3}$$

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where $q = q_0$, $q' = q_{n+1}$, $\varepsilon = \frac{T}{n+1}$. K_j is the short time interval kernel between the space time points $(q_j, j\varepsilon)$ and $(q_j, (j+1)\varepsilon)$:

$$K_j = \exp\left(\frac{i}{\varepsilon}\left[\frac{(q_j - q_{j+1})^2}{2} + (b\varepsilon)^2(2\sin q_j \sin q_{j+1} - 1)\right]\right). \tag{4}$$

Writing the short time interval displacement (in $\varepsilon \to 0$ limit)as

$$\frac{(q_j - q_{j+1})^2}{2} \approx 1 - \cos(q_j - q_{j+1}) \tag{5}$$

we get

$$K_j = e^{\frac{i}{\varepsilon}(1-(b\varepsilon)^2)} P_j, \tag{6}$$

where

$$P_j = \exp(-2ib[\cosh y \cos q_j \cos q_{j+1} + \sinh y \sin q_j \sin q_{j+1}]). \tag{7}$$

The hyperbolic angle y is defined (in $\varepsilon \to 0$ limit) as

$$\cosh y = \frac{1}{2b\varepsilon}, \quad \sinh y \approx \frac{1}{2b\varepsilon}(1 - 2(b\varepsilon)^2). \tag{8}$$

Now we are ready to employ the well known expansion formula [3]

$$P_j = 2\sum_{m=0}^{\infty}\Big[\frac{Ce_{2m}(y)}{p_{2m}} ce_{2m}(q_j) ce_{2m}(q_{j+1}) + \frac{Se_{2m+2}(y)}{s_{2m+2}} se_{2m+2}(q_j) se_{2m+2}(q_{j+1})$$
$$-i\frac{Ce_{2m+1}(y)}{p_{2m+1}} ce_{2m+1}(q_j) ce_{2m+1}(q_{j+1}) - i\frac{Se_{2m+1}(y)}{s_{2m+1}} se_{2m+1}(q_j) se_{2m+1}(q_{j+1})\Big], \tag{9}$$

where $ce_n(x) \equiv ce_n(x, b^2)$ and $se_n(x) \equiv se_n(x, b^2))$ are the even and odd periodic solutions of the Mathieu equation

$$(\frac{d^2}{dx^2} - 2b^2)\cos 2x)y_n = -h_n y_n \tag{10}$$

with h_n being the corresponding eigenvalues. The coefficients in the expansion formula (9) are dependent on the eigenvalues of the Mathieu differential equation [See Appendix].

Inserting (6) and (9) into the full kernel (3) we get

$$K(q, q', T) = \lim_{n\to\infty}\left(\frac{e^{\frac{i}{\varepsilon}(1-(b\varepsilon)^2)}}{\sqrt{2\pi i\varepsilon}}\right)^{n+1}\left(\prod_{j=1}^{n}\int_0^{2\pi} dq_j\right)\prod_{j=0}^{n} P_j. \tag{11}$$

By making use of the the orthogonality relations

$$\int_0^{2\pi} ce_n(x)ce_m(x)dx = \pi\delta_{nm}, \tag{12}$$

$$\int_0^{2\pi} se_n(x)se_m(x)dx = \pi\delta_{nm}, \tag{13}$$

$$\int_0^{2\pi} se_n(x)ce_m(x)dx = 0 \tag{14}$$

we can execute the integration in (11) and arrive at

$$K(q,q',T) = \sum_{m=0}^{\infty}\Omega_m\frac{ce_m(q)ce_m(q')}{\pi} + \sum_{m=1}^{\infty}\Xi_m\frac{se_m(q)se_m(q')}{\pi}, \tag{15}$$

where

$$\Omega_{2m} = \lim_{n\to\infty}(\sqrt{\frac{2\pi}{i\varepsilon}}\frac{Ce_{2m}(y)}{p_{2m}}e^{\frac{i}{\varepsilon}(1-(b\varepsilon)^2)})^{n+1}, \tag{16}$$

$$\Omega_{2m+1} = \lim_{n\to\infty}(-i\sqrt{\frac{2\pi}{i\varepsilon}}\frac{Ce_{2m+1}(y)}{p_{2m+1}}e^{\frac{i}{\varepsilon}(1-(b\varepsilon)^2)})^{n+1} \tag{17}$$

$$\Xi_{2m+2} = \lim_{n\to\infty}(\sqrt{\frac{2\pi}{i\varepsilon}}\frac{Se_{2m+2}(y)}{s_{2m+2}}e^{\frac{i}{\varepsilon}(1-(b\varepsilon)^2)})^{n+1} \tag{18}$$

$$\Xi_{2m+1} = \lim_{n\to\infty}(-i\sqrt{\frac{2\pi}{i\varepsilon}}\frac{Se_{2m+1}(y)}{s_{2m+1}}e^{\frac{i}{\varepsilon}(1-(b\varepsilon)^2)})^{n+1}. \tag{19}$$

Now we can calculate the limits in the above coefficients:

Using the representation for the Mathieu function (with $e^y = \frac{1}{b\varepsilon}$ in $\varepsilon \to 0$ limit)

$$Ce_{2m}(y,b^2) = \frac{p_{2m}}{A_0^{2m}}\sum_{r=0}^{\infty}(-)^r A_{2r}^{2m} J_r(be^{-y})J_r(be^y) \tag{20}$$

and the asymptotic formula for the Bessel functions [4]

$$J_r(x) \approx i^r\frac{e^{-i(x+\frac{r^2-1/4}{2x})}}{\sqrt{-2\pi i x}}, \quad x\to\infty \tag{21}$$

and

$$J_r(x) \approx \frac{x^r}{2^r r!}, \quad x\to 0 \tag{22}$$

we obtain

$$Ce_{2m}(y,b^2) \approx e^{(\frac{i\varepsilon}{8}-\frac{i}{\varepsilon})}\sqrt{\frac{i\varepsilon}{2\pi}}\frac{p_{2m}}{A_0^{2m}}\sum_{r=0}^{\infty}(-i)^r A_{2r}^{2m}\frac{(b^2\varepsilon)^r}{2^r r!}e^{-i\frac{r^2}{2}\varepsilon}. \tag{23}$$

Inserting the above formula into (16) we have

$$\Omega_{2m} = e^{ih_0 T}\lim_{n\to\infty}(\omega_{2m})^{n+1}. \tag{24}$$

Here $h_0 = 1/8 - b^2$ and

$$\omega_{2m} = \frac{1}{A_0^{2m}}\sum_{r=0}^{\infty}(-i)^r A_{2r}^{2m}\frac{(b^2\varepsilon)^r}{2^r r!}e^{-i\frac{r^2}{2}\varepsilon} \approx 1 - i\frac{A_2^{2m}}{A_0^{2m}}\frac{b^2}{2}\varepsilon + O(\varepsilon^2) \tag{25}$$

which from the recurrence relations (A.5) can be written as

$$\omega_{2m} \approx 1 - i\frac{a_{2m}}{2}\varepsilon + O(\varepsilon^2). \tag{26}$$

To obtain the energy spectrum we represent Ω_{2m} as

$$\Omega_{2m} = e^{i(h_0+E_{2m})T} = e^{ih_0 T}\lim_{n\to\infty}(1+\frac{iTE_{2m}}{n+1})^{n+1}. \tag{27}$$

Comparing (27) to (24) (with $T = (n+1)\varepsilon$) and using (26) we get the energy spectrum for the wave functions $ce_{2m}(q)$

$$\mathcal{E}_{2m} = h_0 + E_{2m} = h_0 - \frac{a_{2m}}{2}. \tag{28}$$

In the similar fashions using the representations [3]

$$Ce_{2m+1}(y) = \frac{p_{2m+1}}{A_1^{2m+1}}\sum_{r=0}^{\infty}(-)^r A_{2r+1}^{2m+1}[J_r(be^{-y})J_{r+1}(be^{y}) + J_r(be^{y})J_{r+1}(be^{-y})] \tag{29}$$

$$Se_{2m+1}(y) = -\frac{s_{2m+1}}{iB_1^{2m+1}}\sum_{r=0}^{\infty}(-)^r B_{2r+1}^{2m+1}[J_r(be^{-y})J_{r+1}(be^{y}) - J_r(be^{y})J_{r+1}(be^{-y})] \tag{30}$$

$$Se_{2m+2}(y) = \frac{s_{2m+2}}{iB_2^{2m+2}}\sum_{r=0}^{\infty}(-)^r B_{2r+2}^{2m+2}[J_r(be^{-y})J_{r+2}(be^{y}) - J_r(be^{y})J_{r+2}(be^{-y})] \tag{31}$$

we obtain the remaining coefficients

$$\begin{aligned}
\Omega_{2m+1} &= e^{i(h_0-a_{2m+1}/2)T}, && (32)\\
\Xi_{2m+1} &= e^{i(h_0-b_{2m+1}/2)T}, && (33)\\
\Xi_{2m+2} &= e^{i(h_0-b_{2m+2}/2)T} && (34)
\end{aligned}$$

which exhibit the full energy spectrum.

The coordinate q and time t in the above derivation are dimensionless. To introduce dimension we have to make the following replacements

$$t \to \mu t, \quad q \to \mu q, \tag{35}$$

where μ is the mass of the pendulum.

When the sign of b^2 reversed to $b^2 < 0$ one only needs to replace the argument q of the Mathieu functions by $q = \frac{\pi}{2} + q$, the energy spectrum remains unchanged.

Appendix [3]

$Ce_n(x)$ and $Se_n(x)$ in the formula (9) are called associated Mathieu functions and defined as $Ce_n(x) = ce_n(ix)$ and $Se_n(x) = se_n(ix)$. The constants p_n and s_n are defined as

$$p_{2m} = \frac{ce_{2m}(0)ce'_{2m}(\pi/2)}{A_0^{2m}}, \tag{A.1}$$

$$p_{2m+1} = -\frac{ce_{2m}(0)ce'_{2m}(\pi/2)}{bA_1^{2m+1}}, \tag{A.2}$$

$$s_{2m+1} = \frac{se'_{2m+1}(0)se_{2m+1}(\pi/2)}{bB_1^{2m+1}}, \tag{A.3}$$

$$s_{2m+2} = \frac{se'_{2m+2}(0)se_{2m+2}(\pi/2)}{b^2B_2^{2m+2}}, \tag{A.4}$$

where A_r^n and B_r^n satisfy the following recurrence relations

$$\begin{gathered}
a_{2m}A_0^{2m} - b^2A_2^{2m} = 0\\
(a_{2m} - 4r^2)A_{2r}^{2m} - b^2(A_{2r+2}^{2m} + A_{2r-2}^{2m}) = 0, \quad r \geq 1,
\end{gathered} \tag{A.5}$$

$$\begin{gathered}
(a_{2m+1} - 1 - b^2)A_1^{2m+1} - b^2A_3^{2m+1} = 0\\
(a_{2m+1} - (2r+1)^2)A_{2r+1}^{2m+1} - b^2(A_{2r+3}^{2m+1} + A_{2r-1}^{2m+1}) = 0, \quad r \geq 1
\end{gathered} \tag{A.6}$$

$$(b_{2m+1} - 1 - b^2)B_1^{2m+1} - b^2 B_3^{2m+1} = 0$$
$$(b_{2m+1} - (2r+1)^2)B_{2r+1}^{2m+1} - b^2(B_{2r+3}^{2m+1} + B_{2r-1}^{2m+1}) = 0, \quad r \geq 1 \tag{A.7}$$

$$(b_{2m} - 4)B_2^{2m+1} - b^2 B_4^{2m+2} = 0$$
$$(b_{2m} - 4r^2)B_{2r}^{2m+2} - b^2(B_{2r+2}^{2m+2} - B_{2r-2}^{2m+2}) = 0, \quad r \geq 2 \tag{A.8}$$

and normalization

$$\sum_{r=0}^{\infty} A_{2r}^{2m} > 0, \qquad 2(A_0^{2m})^2 + \sum_{r=0}^{\infty}(A_{2r}^{2m})^2 = 1 \tag{A.9}$$

$$\sum_{r=0}^{\infty} A_{2r+1}^{2m+1} > 0, \qquad \sum_{r=0}^{\infty}(A_{2r+1}^{2m+1})^2 = 1 \tag{A.10}$$

$$\sum_{r=0}^{\infty}(2r+1)B_{2r+1}^{2m+1} > 0, \qquad \sum_{r=0}^{\infty}(B_{2r+1}^{2m+1})^2 = 1 \tag{A.11}$$

$$\sum_{r=0}^{\infty}(2r+2)B_{2r+2}^{2m+2} > 0, \qquad \sum_{r=0}^{\infty}(B_{2r+2}^{2m+2})^2 = 1. \tag{A.12}$$

Here a_{2n}, a_{2n+1}, b_{2n} and b_{2n+1} are the eigenvalues of ce_{2m}, ce_{2n+1}, se_{2n} and se_{2n+1}.

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References

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